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Modulation Spaces with Exponentially Increasing Weights

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Abstract

Modulation spaces were for the first time defined and studied by H. Feichtinger in 1983. Usually the weights used for the definition of modulation spaces have a polynomial growth. But weights with exponential growth can also be used. Such spaces are useful in the study of pseudodifferential operators of infinite order. We shall introduce here a new class of modulation spaces with weights having exponential growth.

Key words: *Modulation space, Gelfand–Shilov-Roumieu space*

Introduction

The modulation spaces were introduced by H. G. Feichtinger in [1]. Much later they were recognized as the right spaces for the time-frequency analysis. They are also useful in the theory of pseudodifferential operators: one can use them to give simple proofs of classical theorems on pseudodifferential operators, such as continuity theorems of Calderon – Vaillancourt type or theorems of composition of pseudodifferential operators.

A comprehensive introduction in the theory of modulation spaces with weights with polynomial growth can be found in [3]. Modulation spaces with submultiplicative weights with exponential growth were studied in [4] and [7]. In [7] the modulation spaces were used to study some classes of infinite order pseudodifferential operators.

In our paper we give a coherent definition of modulation spaces for some classes of weights which are not submultiplicative. These spaces can be used to enlarge the class of infinite order pseudodifferential operators who can be studied using modulation spaces techniques. We shall compare at the end of the next section our classes of modulation spaces with those previously studied.

A Class of Modulation Spaces

As in [5], we shall consider sequences of positive real numbers $(M_p)_p$ who satisfy the following assumptions:

(A1) $M_0 = 1,\ M_1 \geq 1$;

(A2) $M_p^2 \le M_{p-1}M_{p+1}$, $(\forall)p \ge 1$ (logarithmic convexity);

(A3) there exists a constant $H_1 \ge 1$ such that

$$M_{p+q} \le H_1^{p+q} M_p M_q, \ (\forall)p,q \ge 0;$$

(A4) there exists a constant $H_2 \ge 1$ such that

$$\sqrt{p}M_{p-1} \le H_2 M_p, \ (\forall)p \ge 1.$$

It is convenient for our purposes to work with the functions associated to these sequences, $M:(0,\infty)\to[1,\infty)$,

$$M(r) = \sup_{p\ge 0}(p\ln r - \ln M_p), \ (\forall)r > 0.$$

For a, $b > 0$, $(N_p)_p$ a sequence having the same properties as $(M_p)_p$ and N its associated function, we put

$$\|\varphi\|_{a,b} = \left\|\varphi\, \mathrm{e}^{N(b|x|)}\right\|_\infty + \left\|\hat{\varphi}\, \mathrm{e}^{M(a|\xi|)}\right\|_\infty$$

and

$$\mathcal{S}_{a,b}(M,N) = \{\varphi \in \mathcal{S};\ \|\varphi\|_{a,b} < \infty\}.$$

As usual, $\hat{\varphi}$ denotes the Fourier transform of the function φ and, as in [5], for simplicity, φ depends on a single variable. Also, since the sequences $(M_p)_p$ and $(N_p)_p$ are fixed and, consequently, their associated function are, we shall ommit them frequently from the notations.

The spaces $\mathcal{S}_{a,b}(M,N)$ are Banach spaces.Their intersection is a Fréchet space if endowed with the projective limit topology and will be denoted with $\mathcal{S}(M,N)$. The dual of $\mathcal{S}(M,N)$, $\mathcal{S}'(M,N)$, is a space of ultradistributions

In order to introduce our modulation spaces, we have to define first some classes of weighted L^p spaces. We put

$$m_{a,b}:\boldsymbol{R}^2 \to \boldsymbol{R},\ m_{a,b}(x,\xi) = \mathrm{e}^{M(a|\xi|)}\mathrm{e}^{N(b|x|)},\ (\forall)(x,\xi)\in\boldsymbol{R}^2,\ (\forall)a,b>0,$$

$$L^p_{a,b}(= L^p_{m_{a,b}}(\boldsymbol{R}^2) = L^p_{M,N,a,b}(\boldsymbol{R}^2)) = \{F:\boldsymbol{R}^2 \to \boldsymbol{C};\ \iint \left|F(x,\xi)m_{a,b}(x,\xi)\right|^p \mathrm{d}\xi\mathrm{d}x < \infty\}$$

and

$$L^p_{M,N} = \bigcup_{a,b>0} L^p_{a,b}.$$

The spaces $L^p_{a,b}$, $1 \le p < \infty$, is a Banach space with the norm

$$\|F\|_{a,b;p} = \left(\iint \left|F(x,\xi)m_{a,b}(x,\xi)\right|^p \mathrm{d}\xi\mathrm{d}x\right)^{1/p},\ (\forall)F \in L^p_{a,b}.$$

We shall need some simple properties of the space $L^p_{M,N}$.

Lemma 1. The space $L^p_{M,N}$ is invariant under translations.

Proof. If $F \in L^p_{a,b}$ and $(y,\eta)\in\boldsymbol{R}^2$, then

$$\iint \left|F(x-y,\xi-\eta)m_{a/2,b/2}(x,\xi)\right|^p \,\mathrm{d}\xi\mathrm{d}x = \iint \left|F(x,\xi)m_{a/2,b/2}(x+y,\xi+\eta)\right|^p \,\mathrm{d}\xi\mathrm{d}x \leq$$

$$\leq [m_{a,b}(y,\eta)]^p \iint \left|F(x,\xi)m_{a,b}(x,\xi)\right|^p \,\mathrm{d}\xi\mathrm{d}x .$$

Lemma 2 (Hölder inequality). If $F \in L^p_{a,b}$ and $H \in L^q_{(m_{a,b})^{-1}}$, $\frac{1}{p}+\frac{1}{q}=1$, then $FH \in L^1$ and

$$\left|\iint F(x,\xi)H(x,\xi)\,\mathrm{d}x\mathrm{d}\xi\right| \leq \|F\|_{a,b;p}\|H\|_{(m_{a,b})^{-1};p} .$$

This lemma is a direct consequence of the classical Hölder inequality.

Lemma 3. If $F \in L^p_{M,N}$ and $G \in L^1_{M,N}$, then $F * G \in L^p_{M,N}$.

Proof. Let us assume that $F \in L^p_{2a,2b}$, $G \in L^1_{2a,2b}$. Then

$$\left[\iint\left|\iint F(x-y,\xi-\eta)G(y,\eta)\,\mathrm{d}y\mathrm{d}\eta\right|^p m_{a,b}(x,\xi)^p \,\mathrm{d}x\mathrm{d}\xi\right]^{1/p} \leq$$

$$\leq \left[\iint\left(\iint\left|F(x-y,\xi-\eta)\right|m_{2a,2b}(x-y,\xi-\eta)\left|G(y,\eta)\right|m_{2a,2b}(y,\eta)\mathrm{d}y\mathrm{d}\eta\right)^p \mathrm{d}x\mathrm{d}\xi\right]^{1/p} \leq$$

$$\leq \left\|\left|F\right|m_{a,b}\right\|_p \left\|\left|G\right|m_{a,b}\right\|_1 = \|F\|_{a,b:p}\|G\|_{a,b:p} .$$

Definition 1. The modulation space $M_{M,N;p}$ is the space of all the ultradistributions $f \in \mathcal{S}'(M,N)$ such that $V_g f \in L^p_{M,N}$ for some $g \in \mathcal{S}(M,N)$. Here $V_g f$ denotes the short time Fourier transform with window *g* of *f*,

$$V_g f(x,\xi) = \int f(t)\overline{g(t-\xi)}\mathrm{e}^{-2\pi i t\xi}\mathrm{d}t,\ (\forall)(x,\xi) \in \boldsymbol{R}^2 ,$$

where the integral converges in the weak sense.

Our definition extends the definitions given in [3], [4] and [7]. In [3] were defined modulation spaces with polynomially increasing weights. In [4] and [7] exponentially increasing weights of the form $m : \boldsymbol{R}^2 \to \boldsymbol{R},\ m(x,\xi) = \mathrm{e}^{a|\xi|^\gamma}\mathrm{e}^{b|x|^\gamma}$, $(\forall)(x,\xi) \in \boldsymbol{R}^2$, with $\gamma \leq 1$, were considered. This restriction was imposed in order to ensure the submultiplicativity of the weight. Our definition allows to treat also the case when $\gamma \leq 2$ (if $\gamma > 2$ the spaces $\mathcal{S}(M,N)$ are trivial).

The Correctness of the Definition of Modulation Spaces

Lemma 4. If the sequence $(M_p)_p$ satisfies (A1) – (A3) and M is its associated function, then

$$\int_0^\infty \mathrm{e}^{-pM(\varepsilon r)}\,\mathrm{d}r < \infty,\ (\forall)\varepsilon > 0,\ (\forall)p \geq 1 .$$

This lemma follows from lemma 3, section 1, chapter 2 from [8].

Lemma 5. For every $a_1,b_1,a_2,b_2 > 0$ there exist some constants $a,b > 0$ and $C = C(a_1,b_1,a_2,b_2;a,b) > 0$ such that

$$\|V_g\varphi\|_{a,b,p} \leq C\|g\|_{a_1,b_1}\|\varphi\|_{a_2,b_2},\ (\forall)g \in S_{a_1,b_1}(M,N),\ (\forall)\varphi \in S_{a_2,b_2}(M,N) .$$

Proof. If $\varepsilon < \min(a,b)$, then

$$M(ar)+M(\varepsilon r)\le M(H_1 ar),\ N(ar)+N(\varepsilon r)\le N(H_1 ar),\ (\forall)r\ge 0.$$

Therefore, for $g\in \mathcal{S}_{a_1,b_1}(M,N),\ \varphi\in \mathcal{S}_{a_2,b_2}(M,N)$ we have that

$$\left\|V_g\varphi\right\|_{a,b;p}^p=\iint\left|V_g\varphi(x,\xi)\right|^p \mathrm{e}^{pM(a|\xi|)}\mathrm{e}^{pN(b|x|)}\,\mathrm{d}\xi\mathrm{d}x\le$$

$$\le \sup_{(x,\xi)\in\boldsymbol{R}^2}\left[\left|V_g\varphi(x,\xi)\right|\mathrm{e}^{M(H_1a|\xi|)}\mathrm{e}^{N(H_1b|x|)}\right]^p\iint \mathrm{e}^{-pM(\varepsilon|\xi|)}\mathrm{e}^{-pN(\varepsilon|x|)}\,\mathrm{d}\xi\mathrm{d}x.$$

But, accordingly to [6], if

$$a=\frac{1}{2H_1^2}\min(a_1,a_2),\ b=\frac{1}{2H_1^2}\min(b_1,b_2),$$

then there exists some positive constant C such that

$$\sup_{(x,\xi)\in\boldsymbol{R}^2}\left|V_g\varphi(x,\xi)\right|\mathrm{e}^{M(H_1a|\xi|)}\mathrm{e}^{N(H_1b|x|)}\le C\|g\|_{a_1,b_1}\|\varphi\|_{a_2,b_2}.$$

Since from Lemma 4 it follows that

$$\iint \mathrm{e}^{-pM(\varepsilon|\xi|)}\mathrm{e}^{-pN(\varepsilon|x|)}\,\mathrm{d}\xi\mathrm{d}x<\infty,$$

we can take

$$a=\frac{1}{2H_1^2}\min(a_1,a_2),\ b=\frac{1}{2H_1^2}\min(b_1,b_2).$$

The proof is complete.

For $F\in L^p_{M,N}$, $p>1$, and $g\in\mathcal{S}(M,N)$ we define V_g^*F by the formula

$$<V_g^*F,\varphi>=<F,V_g\varphi>,\ (\forall)\varphi\in\mathcal{S}(M,N).$$

Then $V_g^*:L^p_{M,N}\to\mathcal{S}'(M,N)$ is a continuous linear operator.

Proposition 1. If $F\in L^p_{a,b}$ and $g\in\mathcal{S}(M,N)$ then $V_g^*F\in M^p_{a/2,b/2}$.

Proof. Let $g_0\in\mathcal{S}(M,N)$ be a fixed window. Then, as in [3], we can see that

$$\left|V_{g0}V_g^*F(x,\xi)\right|\le C(|F|*|V_{g0}g|)(x,\xi),\ (\forall)(x,\xi)\in\boldsymbol{R}^2.$$

Since $V_{g0}g\in L^p_{a,b}$ also, we obtain, as in the proof of Lemma 3, that there exists a positive constant C such that

$$\left\|V_{g0}V_g^*F\right\|_{a/2,b/2;p}\le C\|F\|_{a,b;p}\left\|V_{g0}g\right\|_{a,b;1}.$$

Proposition 2. The definition of the modulation space $M_{M,N;p}$ is independent of the window $g\in\mathcal{S}(M,N)$.

Proof. Let $g_0, g \in \mathcal{S}(M,N)$, $\|g\|_2 = 1$, and let $f \in M_{M,N;p}$ be such that $V_g f \in L^p_{a,b}$ for some positive constants *a* and *b*. From the inversion formula ([6]) we have that

$$f = V_g^* V_g f .$$

Therefore there exists a positive constant *C*, which does not depend on *f*, such that

$$\left\|V_{g_0} f\right\|_{a/2,b/2;p} = \left\|V_{g_0} V_g^* V_g f\right\|_{a/2,b/2;p} \le C \left\|V_g f\right\|_{a,b;p} \left\|V_{g_0} g\right\|_{a,b;1} .$$

Hence $V_{g_0} f \in L^p_{M,N}$.

Interchanging the roles of *g* and g_0 we see that if $V_{g_0} f \in L^p_{M,N}$, then $V_g f$ belongs also to $L^p_{M,N}$.

The proof is complete.

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Spaţii de modulaţie cu ponderi cu creştere exponenţială

Rezumat

Spaţiile de modulaţie au fost definite şi studiate pentru prima dată de H.Feichtinger în 1983. De regulă, ponderile folosite pentru definirea spaţiilor de modulaţie au o creştere polinomială. Dar pot fi folosite şi ponderi cu creştere exponenţială. Astfel de spaţii sunt utile în studierea operatorilor pseudodiferenţiali de ordin infinit. În această lucrare introducem o nouă clasă de spaţii de modulaţie cu ponderi cu creştere exponenţială.